❖ The sound field of moving sources

Moving point sources

- The pressure field generated by point source of general time and position

$$p'(\mathbf{x},t)=\int\frac{q(\mathbf{y},\tau)}{4\pi|\mathbf{x}-\mathbf{y}|}\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c)d^3\mathbf{y}d\tau$$

- If the source is concentrated at the single moving point, source may be written as

$$q(\mathbf{x},t)=Q(t)\delta(\mathbf{x}-\mathbf{x}_s(t))$$

- So, the pressure field of moving point source is

$$p'(\mathbf{x},t)=\int\frac{Q(\tau)\delta(t-\tau-|\mathbf{x}-\mathbf{x}_s(\tau)|/c)}{4\pi|\mathbf{x}-\mathbf{x}_s(\tau)|}d\tau$$

❖ The sound field of moving sources

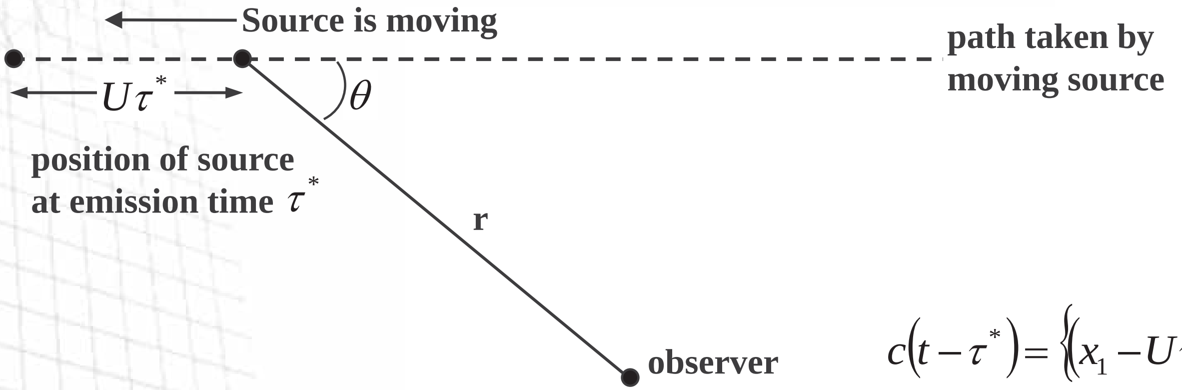
- For the retarded time, τ^*, the pressure field is

$$p'(\mathbf{x},t)=\frac{Q(\tau^*)}{4\pi r\left|1-M_r\right|}$$

where, $\boldsymbol{M_r}$ is relative mach number

and $r=\left|\mathbf{x}-\mathbf{x}_s(\tau^*)\right|$ $c(t-\tau^*)=\left|\mathbf{x}-\mathbf{x}_s(\tau^*)\right|$

- The source is moving with a constant velocity.



$$c(t-\tau^*)=\left\{(x_1-U\tau^*)^2+x_2^2+x_3^2\right\}^{1/2}$$

❖ The sound field of moving sources

- If the source is near the origin at emission time and the observer is far away (| $U\tau^*$ | ≪ | x |), the equation may be rewritten as

$$c\left(t-\tau^*\right)=|\mathbf{x}|\left(1-\frac{x_1}{|\mathbf{x}|}U\tau^*\right)$$

$$\therefore \tau^*=\frac{t-|x|/c}{1-M\cdot\cos\theta}$$

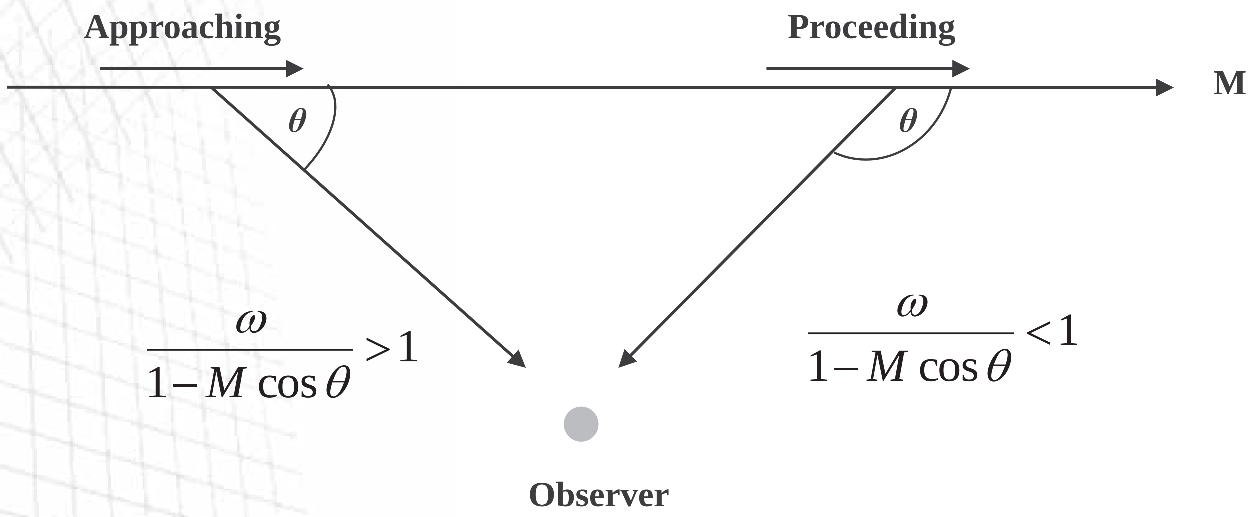
- The frequency of the sound heard by an observer can be determined by comparing p', $\partial p'/\partial t$

$$\frac{\partial p'}{\partial t}=\frac{\partial\tau^*}{\partial t}\frac{\dot{Q}\left(\tau^*\right)}{4\pi r|1-M\cos\theta|} \qquad \frac{1}{p'}\frac{\partial p'}{\partial t}=\frac{1}{1-M\cos\theta}\frac{\dot{Q}\left(\tau^*\right)}{Q\left(\tau^*\right)}$$

❖ The sound field of moving sources

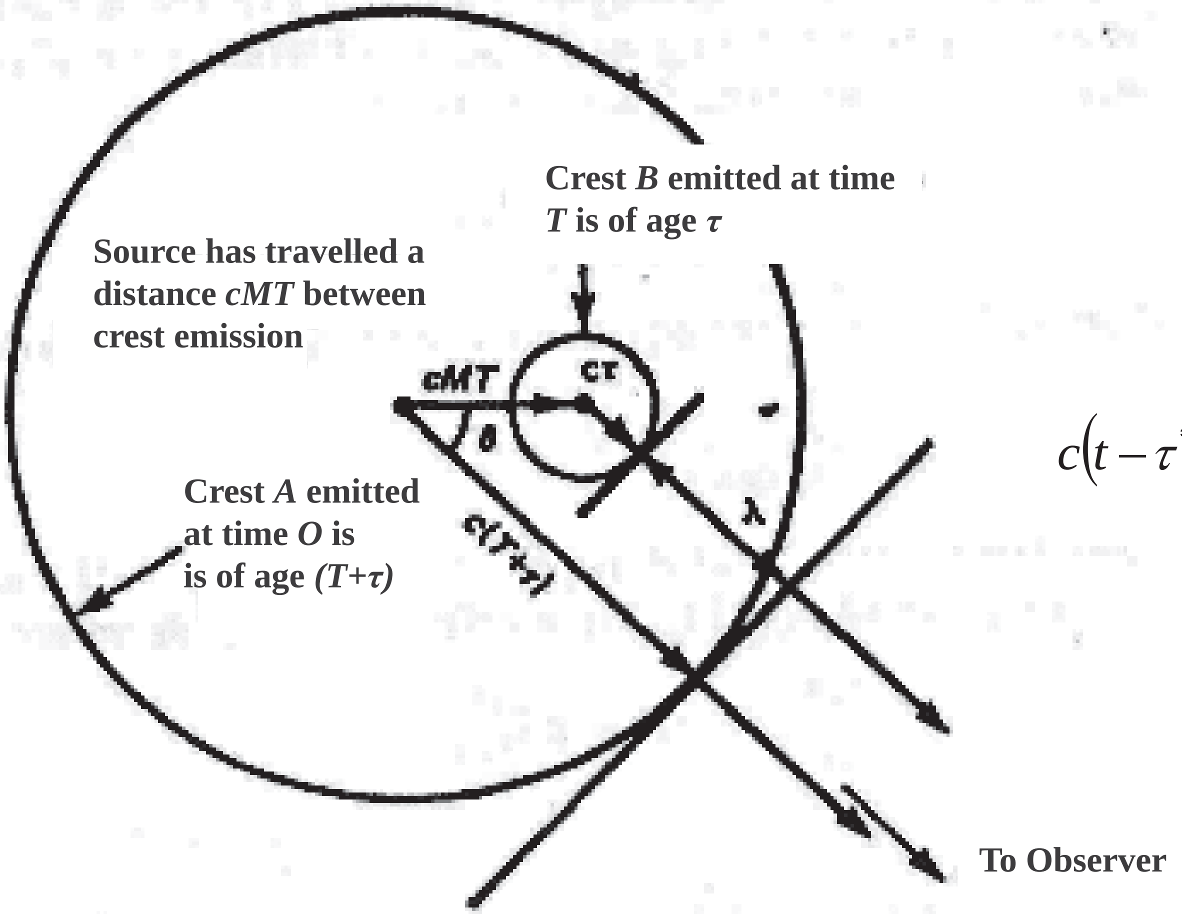
- The sound radiated by a moving source of frequency $\boldsymbol{\omega}$ is heard at **x** at the "Doppler shifted" frequency

$$\omega'\big|_d = \frac{\omega}{1 - M\cos\theta}$$



❖ The sound field of moving sources

- The Doppler factor for a moving source.



$$c\left(t-\tau^*\right)=\left\{\left(x_1-U\tau^*\right)^2+x_2^2+x_3^2\right\}^{1/2}$$

❖ The sound field of moving sources

- To obtain the τ^* , explicitly for any observer position

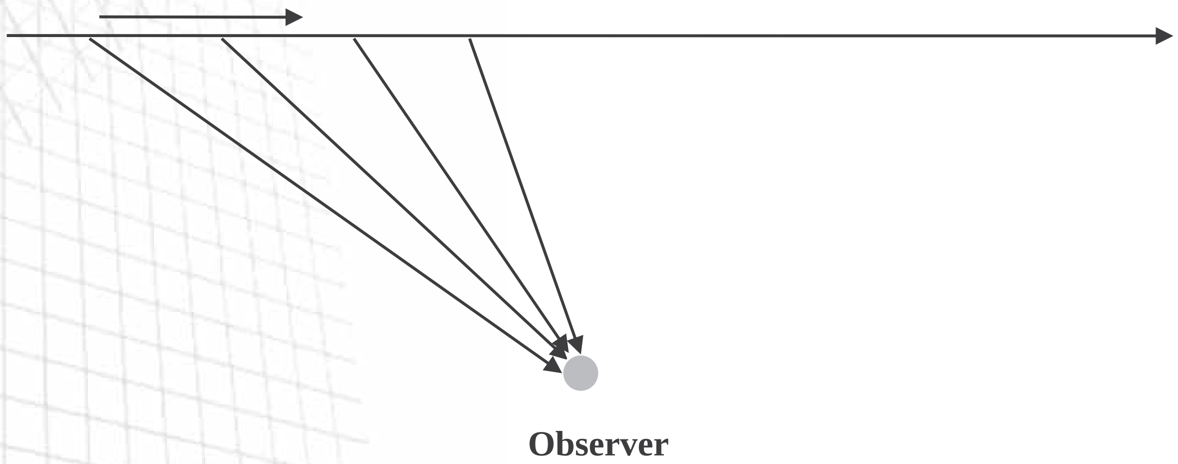
$$\tau^* = \frac{ct - Mx_1 \pm \sqrt{(x_1 - Ut)^2 + (1 - M^2)(x_2^2 + x_3^2)}}{c(1 - M^2)}$$

- Subsonic source velocity

  $\pm \rightarrow -$ sign only (Only one value of ***τ****)

- Supersonic source velocity

  Multiple solutions of ***τ****



❖ The sound field of moving sources

- To convert to the reception time coordinate, two variables are introduced.

$$R = \left\{ (x_1 - Ut)^2 + x_2^2 + x_3^2 \right\}^{1/2} \qquad \Theta = (x_1 - Ut)/R$$

- The retarded time, τ^* , rewritten in reception coordinates

$$\tau^* = t - \frac{R}{c(1-M^2)}\left(M\cos\Theta + \sqrt{1 - M^2 \sin^2\Theta} \right)$$

- The pressure in emission time coordinates is

$$p'(\mathbf{x}, t) = \frac{Q(\tau^*)}{4\pi r |1 - M\cos\theta|}$$

- Rewriting into the reception coordinates

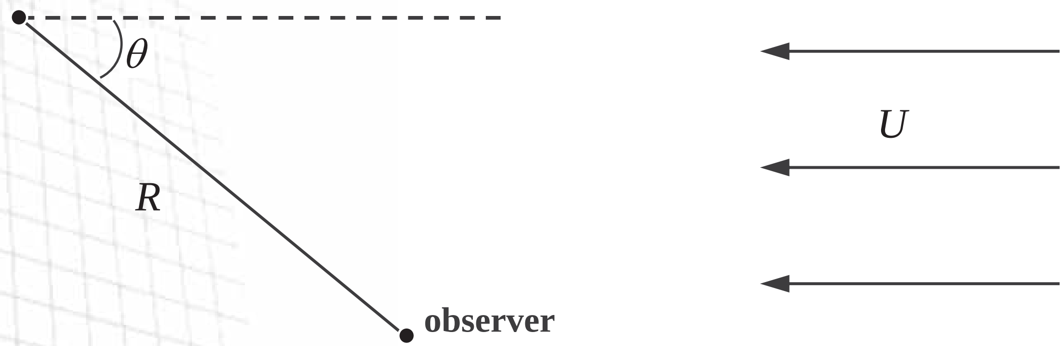
$$r|1 - M\cos\theta| = \left| c(t - \tau^*) - M(x_1 - U\tau^*) \right| = R\sqrt{1 - M^2 \sin^2\Theta}$$

❖ The sound field of moving sources

- And so

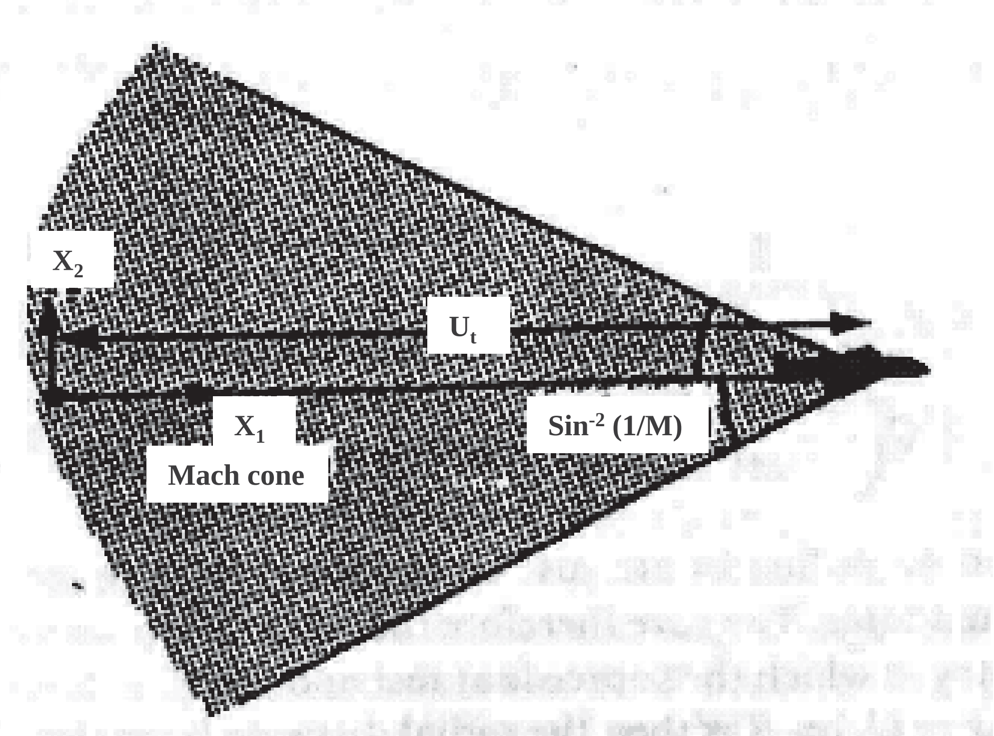
$$p'(\mathbf{x},t) = \frac{1}{4\pi R\left|1 - M^2 \sin^2 \Theta\right|} \times Q\left(t - \frac{R}{c\left(1 - M^2\right)}\left(M \cos \Theta + \sqrt{1 - M^2 \sin^2 \Theta}\right)\right)$$

- Reception coordinates are just coordinates in reference frame that moves with the sources. It is similar to the observation of wind tunnel.



❖ The sound field of supersonic moving sources

- Moving sources
  - In supersonic moving source, the sound is heard within the Mach cone region



$$\left(M^2-1\right)^{\frac{1}{2}}\left(x_2^2+x_3^2\right)^{\frac{1}{2}} < Ut - x_1$$

❖ The sound field of supersonic moving sources

- The sound emitted at two distinct times is heard simultaneous at ***x***

$$\tau^*_{1,2} = \frac{Mx_1 - ct \pm \overline{R}}{c\left(M^2 - 1\right)} \qquad \left(\overline{R} = \sqrt{\left(Ut - x_1\right)^2 - \left(M^2 - 1\right)\left(x_2^2 + x_3^2\right)}\right)$$

- At the observer position, the sound pressure is

$$p(\mathbf{x}, t) = \frac{1}{4\pi\overline{R}}\left\{Q\left(\frac{Mx_1 - ct + \overline{R}}{c\left(M^2 - 1\right)}\right) + Q\left(\frac{Mx_1 - ct - \overline{R}}{c\left(M^2 - 1\right)}\right)\right\}$$

❖ The sound field of supersonic moving sources

- Moving source with finite length
  - To consider source length, the pressure perturbation of the initial position $(\eta_1,0,0)$ is

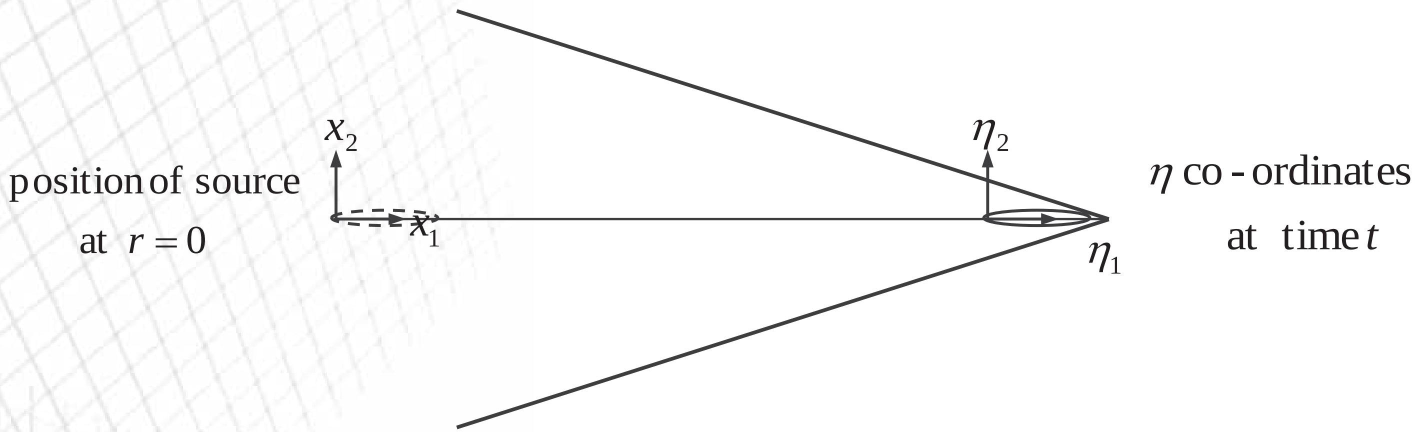
$$p(\mathbf{x},t)=\frac{1}{4\pi\overline{R}}\left\{Q\left(\frac{M(x_1-\eta_1)-ct+\overline{R}}{c(M^2-1)}\right)+Q\left(\frac{M(x_1-\eta_1)-ct-\overline{R}}{c(M^2-1)}\right)\right\}$$

$$\left(\overline{R}=\sqrt{(Ut-x_1+\eta_1)^2-(M^2-1)(x_2^2+x_3^2)}\right)$$

  - For the 1-directional source length l, the source can be considered as a superposition of the moving point sources

$$p(\mathbf{x},t)=\frac{1}{4\pi}\int_{\eta_1=0}^{l}\frac{1}{\overline{R}}\left\{Q\left(\frac{M(x_1-\eta_1)-ct+\overline{R}}{c(M^2-1)}\right)+Q\left(\frac{M(x_1-\eta_1)-ct-\overline{R}}{c(M^2-1)}\right)\right\}d\eta_1$$

❖ The sound field of supersonic moving sources



- If the source has a finite length in the 1-direction, the sound heard is that due to an integral of the source element, and retarded time is a function of source element

$$\tau = \frac{M(x_1-\eta_1) - ct \pm \eta_1^{\frac{1}{2}}(2\sigma)^{\frac{1}{2}}}{c(M^2-1)} \qquad \frac{d\tau^*}{d\eta_1} = \frac{x_1-\eta_1-U\tau^*}{cr(1-M_r)}$$

❖ The sound field of supersonic moving sources

- For directions in which the effective wavelength is much longer than the body dimension, the effect of the variation of retarded time along the source length can be negligible

- For an observer on the Mach cone, $M_r \sim 1$, retarded time varies rapidly along the source length. A distant observer hears the accumulated sound emitted by the source during the entire time.

- To demonstrate clearly, pressure perturbation is calculated in the below conditions.

$$\sigma = Ut - x_1 \qquad \bar{R} = \eta_1^{\frac{1}{2}}\left(2\sigma + \eta_1\right)^{\frac{1}{2}}$$

$$\eta_1 = 0 \;\Big|\; \bar{R} \text{ is singular}$$

$$\bar{R}_{\text{approx.}} = \eta_1^{\frac{1}{2}}\left(2\sigma\right)^{\frac{1}{2}} \quad \text{for } \sigma >> l$$

$$p'(x,t) = \frac{1}{4\pi(2\sigma)^{\frac{1}{2}}}\int_{\eta_1=0}^{l}\left\{Q\left(\eta_1, \frac{M(x_1-\eta_1)-ct+(2\sigma)^{\frac{1}{2}}\eta_1^{\frac{1}{2}}}{c(M^2-1)}\right) + Q\left(\eta_1, \frac{M(x_1-\eta_1)-ct-(2\sigma)^{\frac{1}{2}}\eta_1^{\frac{1}{2}}}{c(M^2-1)}\right)\right\}\frac{d\eta_1}{\eta_1^{\frac{1}{2}}}$$

❖ The sound field of supersonic moving sources

- For a source with finite life-span, the pressure perturbation is not decayed by squared root of ***σ*** in the case of arbitrarily ***σ***. The terms of in the integrand are to be evaluated at retarded time.

$$\tau = \frac{M\left(x_1 - \eta_1\right) - ct \pm \eta_1^{\frac{1}{2}}\left(2\sigma\right)^{\frac{1}{2}}}{c\left(M^2 - 1\right)}$$

- If the source emits sound for a finite time ***0< τ <T***, for an observer in the very distant far-field, the integrand is zero over the most ranges of integration. Since the finite time duration time is crucial thing.

$$\frac{\partial \tau}{\partial \eta_1} = \frac{-M \pm \frac{1}{2}\eta_1^{-\frac{1}{2}}\left(2\sigma\right)^{\frac{1}{2}}}{c\left(M^2 - 1\right)}$$

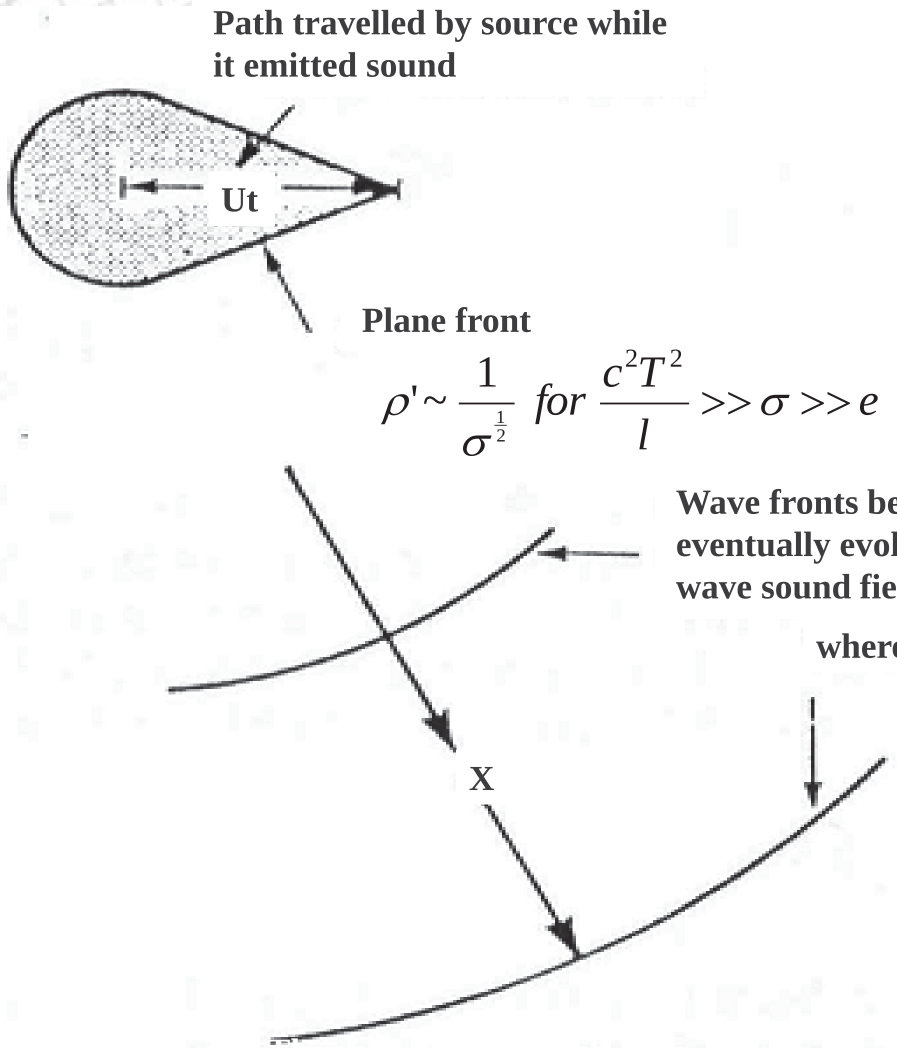
❖The sound field of supersonic moving sources

- Mach wave sound vanishes unless $Mx_1=ct$. $Mx_1=ct$ and $Ut-x_1 = \sigma$ combine to to give |x|=ct, the Mach wave sound is

$$p'(x,t) \xrightarrow[|\mathbf{x}|\to\infty]{} \frac{c(M^2-1)}{4\pi\sigma}\int_0^T Q(\eta_1(\tau),\tau)d\tau$$
$$= \frac{U}{4\pi|\mathbf{x}|}\int_0^T Q(\eta_1(\tau),\tau)d\tau$$

- The Mach wave sound heard in the very far-field decays inversely with distance. All the sound ever released during the entire history of the source is heard by the distant observer in one big bang!

❖The sound field of supersonic moving sources



$$\rho' \sim \frac{1}{\sigma^{\frac{1}{2}}} \quad for \ \frac{c^2T^2}{l} >> \sigma >> e$$

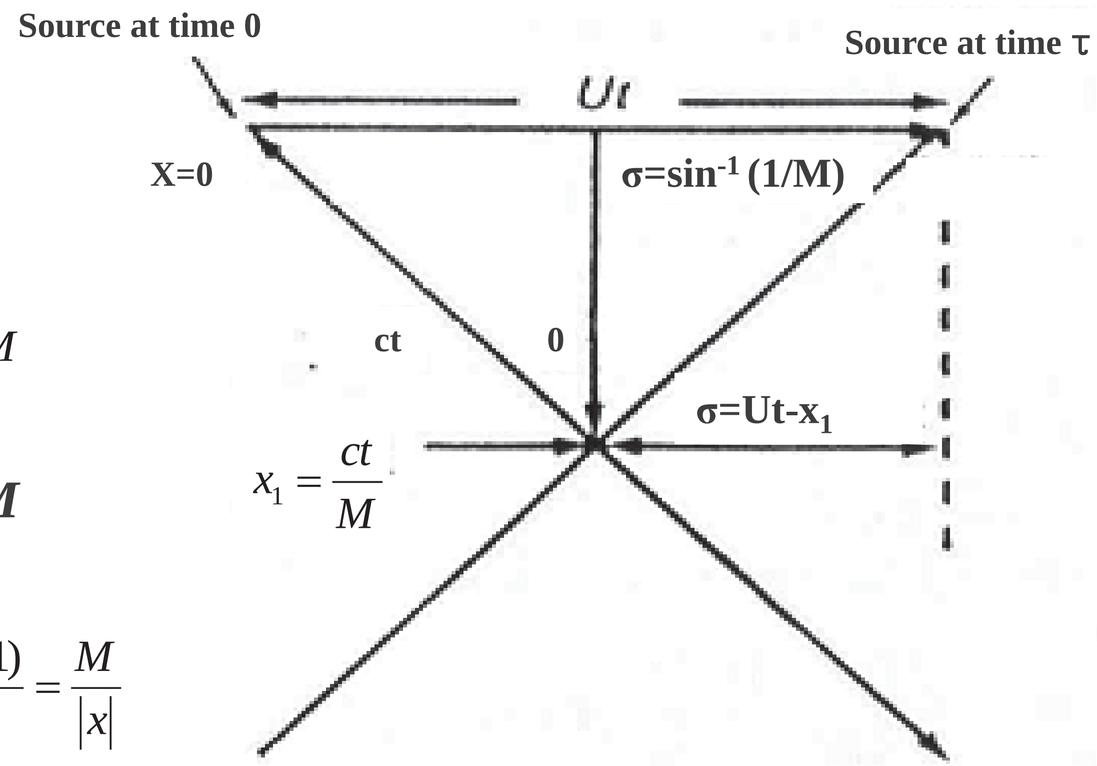
$$\rho^1 \sim \frac{(M^2-1)}{\sigma} = \frac{M}{|x|} \quad for \ \sigma >> \frac{c^2T^2}{l}$$

❖ The sound field of supersonic moving sources

$$|x| = ct = \frac{Ut-\sigma}{\sin\theta} = (Ut-\sigma)M$$

That is, ***ct(1-M²)=-σM***

$$\sigma = |x|\frac{(M^2-1)}{M} \quad and \quad \frac{(M^2-1)}{\sigma} = \frac{M}{|x|}$$



The Mach wave geometry

❖ The sound field of supersonic moving sources

● Moving sources with mass injected and force applied

- The sound field generated when the field of density(***ρ_0***) is injected at a rate ***$\rho_0\, \beta'(t)$*** and a force ***f(t)*** applied at the moving point ***x=Ut***

- Mass conservation : $$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} = \rho_0 \dot{\beta} \delta(\vec{x} - \vec{U}t)$$

- Linear momentum : $$\rho_0 \frac{\partial \vec{v}}{\partial t} + \nabla p' = \vec{f} \delta(\vec{x} - \vec{U}t)$$

- Combining and Solution

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \rho_0 \frac{\partial}{\partial t} \dot{\beta} \delta(\vec{x} - \vec{U}t) - \frac{\partial}{\partial x_i}\{ f_i \delta(\vec{x} - \vec{U}t)$$

$$p'(\vec{x}, t) = \rho_0 \frac{\partial}{\partial t}\left[\frac{\dot{\beta}(\tau^*)}{4\pi r |1 - M_r|}\right] - \frac{\partial}{\partial x_i}\left[\frac{f_i(\tau^*)}{4\pi r |1 - M_r|}\right]$$

❖ The sound field of supersonic moving sources

- but $$\frac{\partial \tau^*}{\partial x_i} = \frac{-(x_i - U_i \tau^*)}{cr(1-M_r)}$$

- Hence $$p'(\vec{x},t) = \left[\rho_0 \ddot{\beta}(\tau^*) + \dot{f}_i(\tau^*)\right] \frac{1}{4\pi r(1-M_r)\left|1-M_r\right|}$$

- Effect of source motion ~ $$\frac{1}{(1-M_r)^2}$$

● Note

- ***β(τ)*** and ***f(τ)*** are not independent in general

❖ The sound from the moving foreign bodies

- The foreign bodies in the flow

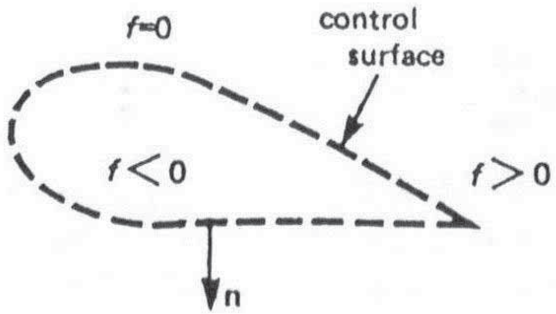


Fig. 9.11 — Definition of the function $f(x, t)$.

f **is sufficiently smooth**

$$\nabla f = \vec{n}\left|\nabla f\right|$$

$$\frac{\partial f}{\partial t} = u_i \frac{\partial f}{\partial x_i} = 0 \quad on\ c.s.$$

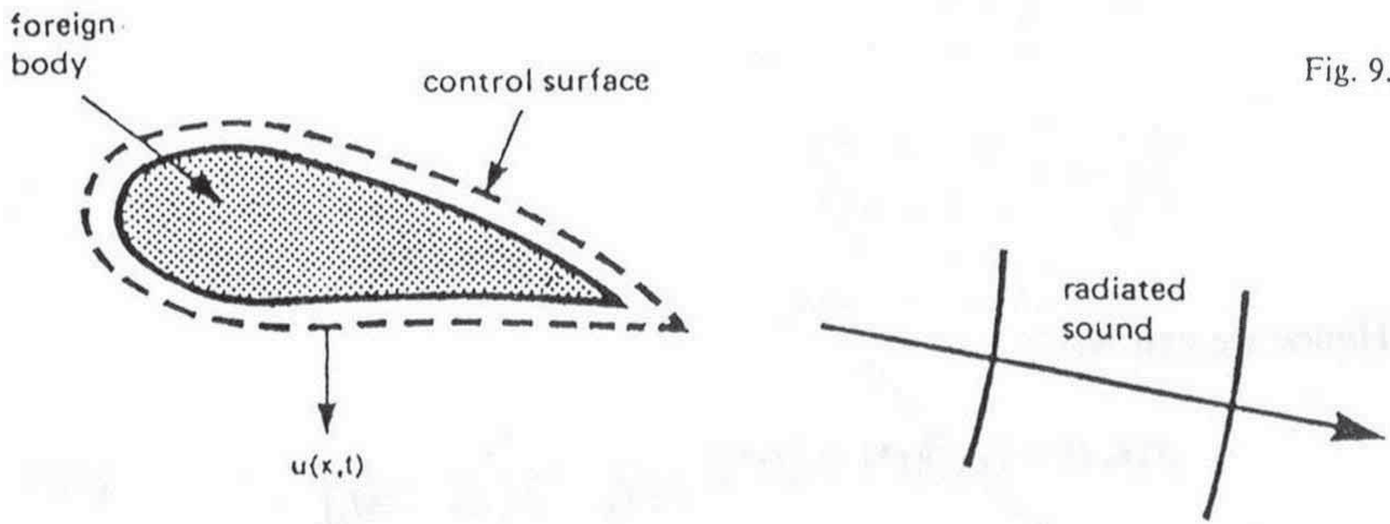


Fig. 9.10 — A foreign body enclosed by a control surface which moves with velocity $u(x, t)$.

❖ The sound from the moving foreign bodies

- Delta function ***δ(f) ≠ 0*** on ***f = 0***

$$\int_{\infty} q(\vec{x})\delta(f)d^3\vec{x} = \int_s \frac{q(\vec{x})}{|\nabla f|}ds$$

- Define Heaviside function ***H(x)***

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{cases}$$

- Inside the control surface, H(f)=0 , where f < 0

$$H(f)\left\{\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i\right)\right\} = 0$$

❖ The sound from the moving foreign bodies

- Mass conservation equation

$$\frac{\partial}{\partial t}\left(H\rho'\right)+\frac{\partial}{\partial x_i}\left(H\rho v_i\right)=\rho'\frac{\partial H}{\partial t}+\rho v_i\frac{\partial H}{\partial x_i}=\left(\rho-\rho_0\right)\frac{\partial H}{\partial t}+\rho v_i\frac{\partial H}{\partial x_i}$$

$$=\left(\rho-\rho_0\right)\frac{\partial f}{\partial t}\delta(f)+\rho v_i\frac{\partial f}{\partial x_i}\delta(f)$$

$$=\left\{\underline{\rho_0 u_i}+\underline{\rho(v_i-u_i)}\right\}\frac{\partial f}{\partial x_i}\delta(f)$$

Due to the motion of body **Due to the fluid motion**

- Momentum equation

$$\frac{\partial}{\partial t}\left(H\rho v_i\right)+\frac{\partial}{\partial x_j}\left(Hp_{ij}+H\rho v_i v_j\right)=\left\{\rho v_i(v_j-u_j)+p_{ij}\right\}\frac{\partial f}{\partial x_j}\delta(f)$$

❖ The sound from the moving foreign bodies

- Combining mass & momentum equation

$$\frac{\partial^2}{\partial t}(H\rho') - c^2\nabla^2(H\rho') = \frac{\partial^2(HT_{ij})}{\partial x_i x_j} - \frac{\partial}{\partial x_i}\left(\{\rho v_i(v_j - u_j) + p_{ij}\}\frac{\partial f}{\partial x_j}\delta(f)\right) + \frac{\partial}{\partial t}\left(\{\rho(v_i - u_i) + \rho_o u_i\}\frac{\partial f}{\partial x_i}\delta(f)\right)$$

$$where,\ T_{ij} = \rho v_i v_j + p_{ij} - c^2(\rho - \rho_0)\delta_{ij}$$

- The solution is

$$4\pi c^2 H\rho'(x,t) = \frac{\partial^2}{\partial x_i x_j}\int \frac{HT_{ij}}{|x-y|}\delta(t-\tau-|x-y|/c)d^3y d\tau - \frac{\partial}{\partial x_i}\int \frac{\{\rho v_i(v_j - u_j) + p_{ij}\}}{|x-y|}\frac{\partial f}{\partial y_j}\delta(f)\delta(t-\tau-|x-y|/c)d^3y d\tau + \frac{\partial}{\partial t}\int \frac{\{\rho(v_i - u_i) + \rho_o u_i\}}{|x-y|}\frac{\partial f}{\partial y_j}\delta(f)\delta(t-\tau-|x-y|/c)d^3y d\tau\left(\frac{\partial f}{\partial x_i}\delta(f)\right)$$

❖The sound from the moving foreign bodies

- To consider the effect of moving solid boundary, it is convenient to introduce a moving frame with acceleration, ***a***, velocity ***V***, at any fixed point , ***η***

$$\mathrm{V} = \left.\frac{\partial \mathrm{y}(\eta,\tau)}{\partial \tau}\right|_{\eta=\mathrm{Constant}}, \quad \mathrm{a} = \left.\frac{\partial \mathrm{V}(\eta,\tau)}{\partial \tau}\right|_{\eta=\mathrm{Constant}}$$

- And, moving surface is stationary relative to the moving frame.
- The previous monopole term of sound generated by the stationary foreign body is added moving effect

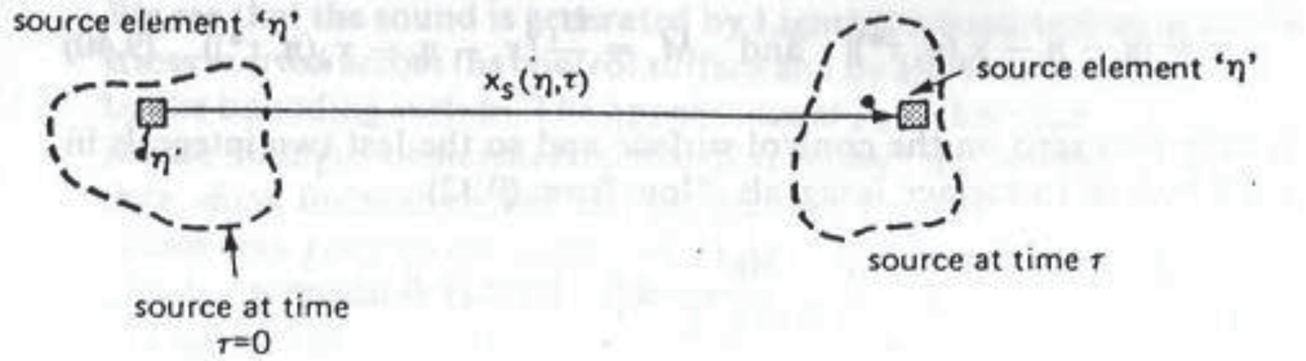
$$\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int_S\left[\frac{\rho\mathbf{v}\cdot\mathbf{n}}{R}\right]dS = \frac{1}{4\pi c^2}\int\int_S\frac{\rho\mathbf{v}\cdot\mathbf{n}}{R}\frac{\partial}{\partial\tau}\delta\left(t-\tau-\frac{R}{c}\right)dSd\tau$$

$$\longrightarrow \quad -\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int\int_{V(\tau)}\frac{\rho a_j}{R}\delta\left(t-\tau-\frac{R}{c}\right)d\mathbf{V}d\tau$$

$$+\frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i\partial x_j}\int\int_{V(\tau)}\frac{\rho v_i v_j}{R}\delta\left(t-\tau-\frac{R}{c}\right)d\mathbf{V}d\tau$$

❖ The sound from the moving foreign bodies

- To obtain more generality, the foreign body is translating and rotating



$$V(\eta,\tau) = V_0(\tau) + \Omega(\tau) \times \eta \qquad y = \eta + x_s(\eta,\tau)$$

- The integration of volume and surface is independent of the retarded time. Hence, the source terms of quadrupole and dipole are not changed.

❖ The sound from the moving foreign bodies

$$\rho'(\mathbf{x},t)=\frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int\int_{V(t_0)}\left[\frac{T_{ij}}{R}\right]d\eta d\tau-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\int\int_{S(t_0)}\left[\frac{f_i}{R}\right]d\eta d\tau$$

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int\int_{V_c(t_0)}\left[\frac{\rho a_j}{R}\right]d\eta d\tau+\frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int\int_{V_c(t_0)}\left[\frac{\rho v_i v_j}{R}\right]d\eta d\tau$$

- Using the identity of function '***g***', it is carried out for the integration with respect to the retarded time.

$$\int_{-\infty}^{\infty} f(\tau)\delta[g(\tau)]d\tau=\sum_i \frac{f(\tau_e^i)}{\left|\frac{dg(\tau_e^i)}{d\tau_e}\right|}$$

$$\left(\frac{\partial g}{\partial \tau}\right)_\zeta=1-\frac{\mathbf{R}}{c_0 R}\cdot\left(\frac{\partial \mathbf{y}}{\partial \tau}\right)_\zeta=1-\frac{\mathbf{R}}{R}\cdot\mathbf{M}$$

❖ The sound from the moving foreign bodies

- The Ffowcs Williams-Hawkings equation is derived.

$$\rho'(\mathbf{x},t) = \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V(t_0)}\left[\frac{T_{ij}}{R\left|1-(\mathbf{R}/R)\cdot\mathbf{M}\right|}\right]_{\tau=\tau_e} d\eta$$

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\int_{S(t_0)}\left[\frac{f_i}{R\left|1-(\mathbf{R}/R)\cdot\mathbf{M}\right|}\right]_{\tau=\tau_e} d\eta$$

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int_{V_c(t_0)}\left[\frac{\rho a_j}{R\left|1-(\mathbf{R}/R)\cdot\mathbf{M}\right|}\right]_{\tau=\tau_e} d\eta$$

$$+\frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V_c(t_0)}\left[\frac{\rho v_i v_j}{R\left|1-(\mathbf{R}/R)\cdot\mathbf{M}\right|}\right]_{\tau=\tau_e} d\eta$$

- The first term of 'FW-H' equation corresponds to the solution that arises in Lighthill's theory. And, the second term represents the sound generated by fluctuating force, $\boldsymbol{f_i}$, exerted by solid boundary. And, the remaining two terms means the sound generated by the volume displacement effects

❖ The sound from the moving foreign bodies

- If the velocity V of any point of source region is supersonic, the Doppler factor comes $$1-\frac{\mathbf{R}}{R}\cdot\mathbf{M}=1-M\cos\theta$$
- It vanishes at the angle, $$\theta=\cos^{-1}\frac{1}{M}$$
- The resulting singularities are the same as those that were associated with Mach wave emission. However, when supported by foreign bodies Mach waves are often able to coalesce into the intense shock.
- When the surface, S, is stationary, ***a≡M=V=0, η=y***, FW-H equation reduces to Curle's equation

$$\rho'(\mathbf{x},t)=\frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i\partial x_j}\int_V\frac{T_{ij}}{R}\left(\mathbf{y},t-\frac{R}{c}\right)d^3\mathbf{y}-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\int_S\frac{f_i}{R}\left(\mathbf{y},t-\frac{R}{c}\right)dS$$

❖ The sound from the moving foreign bodies

- Using moving coordinate system ***y* = *η* + *x*$_s$(*η*,*τ*)**

$$4\pi c^2 H\rho'(x,t) = \frac{\partial^2}{\partial x_i x_j}\int_V \frac{JT_{ij}}{r|1-M_r|} d^3\eta$$
$$-\frac{\partial}{\partial x_i}\int_S \frac{\{\rho v_i (v_j - u_j) + p_{ij}\}}{r|1-M_r|} n_j K dS(\eta)$$
$$+\frac{\partial}{\partial t}\int_S \frac{\{\rho (v_i - u_i) + \rho_o u_i\}}{r|1-M_r|} n_j K dS(\eta)$$

- Where ***J,K*** are Jacobians, if no volume change, ***J=K=1***

❖ The sound from the moving foreign bodies

- If the surface is impenetrable, the normal surface velocity must be equal to that of the flow. (***u·n = v·n***)

$$4\pi c^2 H\rho'(x,t) = \frac{\partial^2}{\partial x_i x_j}\int_V \frac{JT_{ij}}{r\left|1-M_r\right|}d^3\eta$$ ← **Quadrupole**

$$-\frac{\partial}{\partial x_i}\int_S \frac{p_{ij}n_j K}{r\left|1-M_r\right|}dS(\eta)$$ ← **Dipole**

$$+\frac{\partial}{\partial t}\int_S \frac{\rho_o \vec{v}\vec{n}K}{r\left|1-M_r\right|}dS(\eta)$$ ← **Monopole**

❖ The sound from the moving foreign bodies

- A compact pulsating sphere moving at low mach number

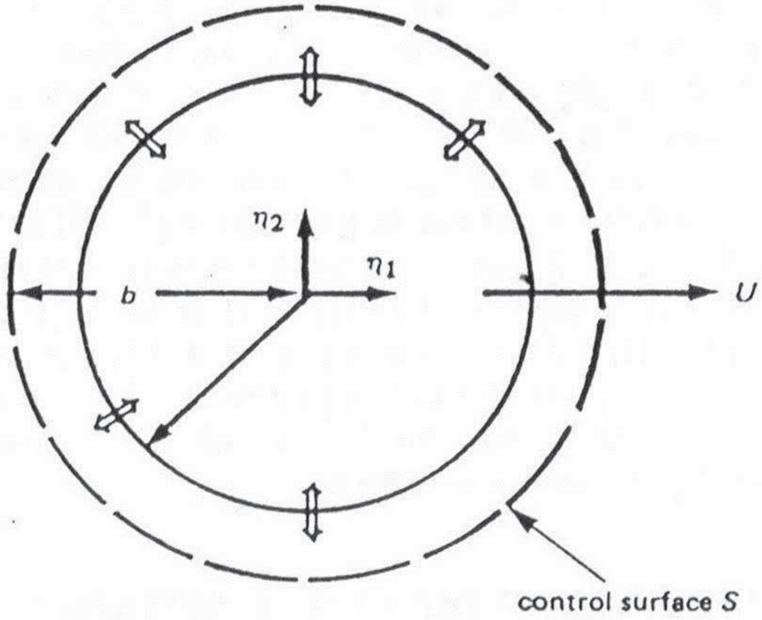


Fig. 9.13 — A moving pulsating sphere.

· $M^2 \ll 1$ negligible

· Linear & compact pulsation, $\omega a/c \ll 1$

$$4\pi c^2 H\rho'(x,t) = \frac{\partial^2}{\partial x_i x_j}\int_V \frac{T_{ij}}{r\left|1-M_r\right|}d^3\vec{\eta}$$

$$-\frac{\partial}{\partial x_i}\int_S \frac{\left\{\rho v_i(v_j-u_j)+p_{ij}\right\}}{r\left|1-M_r\right|}n_j ds(\vec{\eta})$$

$$+\frac{\partial}{\partial t}\int_S \frac{\rho(v_i-u_i)+\rho_o u_i}{r\left|1-M_r\right|}n_j ds(\vec{\eta})$$

❖ The sound from the moving foreign bodies

- For an observer in the for-field, equation with the retarded time
  - $c(t-\tau^*(\eta))=|x-\eta-U\tau^*|$
  - $\tau^* = (t - |x|/c + x_i n_i / |x|c) \times 1/M\cos\theta$

- since the source is compact, and **M²<<1** (incompressibel), the field close to the body satisfies the potential equation
  - $\nabla^2\Phi=0$ --------(*)

- On $\eta=|\eta|=A$, the normal velocity of the sphere
  - $n\cdot\nabla\varphi=\dot{A}(\tau)+U\eta_1/\eta$

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- The solution of eq.(∗) with this BC. Becomes

$$\varphi = -\frac{\dot{A}(\tau)a^2}{\eta} - \frac{1}{2}UA^3\frac{\eta_1}{\eta^3}$$

  - Velocity $\vec{V} = \frac{\partial\varphi}{\partial y_i} = \frac{\partial\varphi}{\partial\eta_i} = \dot{A}(\tau)a^2\frac{\eta_i}{\eta^3} - \frac{1}{2}UA^3(\tau)\{\frac{\delta_{i1}}{\eta^3} - \frac{\eta_i\eta_1}{\eta^5}\}$

  - Pressure perturbation form unsteady Bernoulli

$$p' = -\rho_0\left.\frac{\partial\varphi}{\partial t}\right|_\eta - \frac{1}{2}\rho v^2 = -\rho_0\left.\frac{\partial\varphi}{\partial t}\right|_\eta + \rho_0 U\frac{\partial\varphi}{\partial\eta_1} -- \frac{1}{2}\rho v^2$$

- Let’s evaluate each term;
  - monopole

$$\frac{\partial}{\partial t}\int_S \frac{\rho(v_i - u_i) + \rho_o u_i}{r|1-M_r|} n_i dS(\vec{\eta}) = \frac{\partial}{\partial t}\int_S \rho_0\frac{\left\{\dot{A}(\tau^*) + UA^3(\tau^*)\eta_1/b^4\right\}}{r|1-M_r|}dS(\vec{\eta}) \qquad ------(**)$$

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- Expand with ***τ****,

$$\dot{A}(\tau^*)+UA^3(\tau^*)\frac{\eta_1}{b^4}=\dot{A}(\tau_0^*)+UA^3(\tau_0^*)\frac{\eta_1}{b^4}+3U\dot{A}(\tau_0^*)+UA^3(\tau_0^*)\frac{\eta_1\eta_i}{a^2}\frac{x_i}{|x|\cdot c(1-M\cos\theta)}+HOT$$

  - Where, $\tau_0^*=\dfrac{t-|x|/c}{1-M\cos\theta}\quad(\vec{\eta}\to 0)$

- Evaluating the integral (∗∗) for large |**x**|,

$$\frac{\partial}{\partial t}\int_S \frac{\rho(v_i-u_i)+\rho_o u_i}{r|1-M_r|}n_i dS(\vec{\eta})$$

$$=\frac{\partial}{\partial t}\left\{\frac{4\pi a^2\rho_0\dot{A}(\tau_0^*)}{|x|\cdot c(1-M\cos\theta)^2}\right\}$$

$$\approx\frac{4\pi a^2\rho_0\ddot{A}(\tau_0^*)}{r\cdot c(1-M\cos\theta)^3}$$

$$|x|>>|\vec{U}\tau*|$$

$$\frac{1}{r|1-M_r|}=\frac{1}{|x|(1-M\cos\theta)}$$

$$\frac{\partial\tau_0^*}{\partial t}=\frac{1}{1-M\cos\theta}$$

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- Similarly
  - Dipole

$$-\frac{\partial}{\partial x_i}\int_S \frac{\{\rho v_i(v_j-u_j)+p_{ij}\}}{r(1-M_r)}n_j ds = \frac{2\pi a^2\rho_0 M\cos\theta}{|x|}\ddot{A}(\tau_0^*)$$

  - Quadrupole

$$\frac{\partial^2}{\partial x_i x_j}\int_V \frac{T_{ij}}{r(1-M_r)}d^3\eta \approx O\left(\frac{\rho_0 a^2 M^2\ddot{A}}{|x|}\right)$$

- In total,

$$c^2\rho'(\vec{x},t) = \frac{\rho_0 a^2\ddot{A}(\tau_0^*)}{|x|(1-M\cos\theta)^{3\frac{1}{2}}}$$

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- Note

  - motion amplifies the pressure perturbation by 3½ Doppler factor, far more complicated than the point source case(linear case)

  - It is due to the coupled momentum flax associated with a volume flux.

  - The sound field generated by the force is only a mach number smaller than the leading term!!